Lecture 4.3: Closures and Equivalence Relations

CS 250, Discrete Structures, Fall 2011

Nitesh Saxena

*Adopted from previous lectures by Cinda Heeren

Course Admin

- Mid-Term 2 Exam Graded
  - Solution has been posted
  - Any questions, please contact me directly
  - Will distribute today
- HW3 Graded
  - Solution has been posted
  - Please contact the TA for questions
  - Will distribute today

Final Exam

- Thursday, **December 8**, **10:45am-1:15pm**, lecture room
  - Heads up!
  - Please mark the date/time/place

- Our last lecture will be on December 6
  - We plan to do a final exam review then

HW4

- Expect HW4 to be posted later this week
- Covers the chapter on Relations
- Will be due in 10 days from the day of posting

Outline

- Closures
- Equivalence Relations

N-ary Relations

- So far, we were talking about binary relations – defined on two sets.
- Can be generalized to N sets
- Ex: $R = \{(a, b, c): a < b < c\}$, defined on set of integers – a 3-ary relation
- Applications in databases

Closure

- Consider relation R={(1,2),(2,2),(3,3)} on the set A = {1,2,3,4}.
- Is R reflexive? No
- What can we add to R to make it reflexive? (1,1), (4,4)

R' = R U {(1,1),(4,4)} is called the *reflexive closure* of R.

Closure

- Definition:

The closure of relation R on set A with respect to property P is the relation R′ with

1. $R \subseteq R'$
2. R′ has property P
3. $\forall S$ with $R \subseteq S$ and S has property P, $R' \subseteq S$.

Reflexive Closure

- Let r(R) denote the reflexive closure of relation R.

  Then r(R) = R U { (a,a): $\forall$ a $\in$ *A* }

- Fine, but does that satisfy the definition?
  1. R $\subseteq$ r(R) We added edges!
  2. r(R) is reflexive By defn
  3. Need to show that for any S with particular properties, r(R) $\subseteq$ S.

  Let S be such that R $\subseteq$ S and S is reflexive. Then {(a,a): $\forall$ a $\in$ *A* } $\subseteq$ S (since S is reflexive) and R$\subseteq$S (given). So, r(R) $\subseteq$ S.

Symmetric Closure

- Let $s(R)$ denote the symmetric closure of relation R.

  Then $s(R) = R \cup \{$ (b,a): (a,b) $\in$ R $\}$

- Fine, but does that satisfy the definition?
  1. $R \subseteq s(R)$ We added edges!
  2. $s(R)$ is symmetric By defn
  3. Need to show that for any S with particular properties, $s(R) \subseteq S$.

  Let S be such that $R \subseteq S$ and S is symmetric. Then $\{(b,a): (a,b) \in R\} \subseteq S$ (since S is symmetric) and $R \subseteq S$ (given). So, $s(R) \subseteq S$.

Transitive Closure

- Let t(R) denote the transitive closure of relation R.

  Then t(R) = R U { (a,c): $\exists$b (a,b),(b,c) $\in$ R }

- Example: A={1,2,3,4}, R={(1,2),(2,3),(3,4)}.

Apply definition to get:

t(R) = {(1,2),(2,3),(3,4), (1,3), (2,4) }

Which of the following is true:

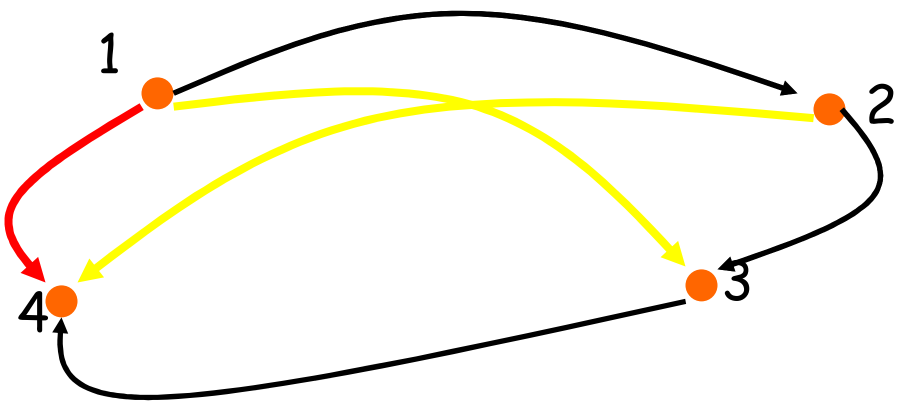
a) This set is transitive, but we added too much.
b) This set is the transitive closure of R.
c) This set is not transitive, one pair is missing.
d) This set is not transitive, more than 1 pair is missing.

Transitive Closure

- So how DO we find the transitive closure?

Draw a graph.

- Example: $A=\{1,2,3,4\}$, $R=\{(1,2),(2,3),(3,4)\}$.



- Define a *path* in a relation R, on A to be a sequence of elements from A: $a, x_1, \ldots x_i, \ldots x_{n-1}, b$, with $(a, x_1) \in R$, $\forall i\ (x_i, x_{i+1}) \in R$, $(x_{n-1}, b) \in R$.

"Path from a to b."

Transitive Closure

- Formally:

  If t(R) is the transitive closure of R, and if R contains a path from a to b, then (a,b) ∈ t(R)

A technique:

- For a set R consisting of n elements, t(R) can be specified by the matrix: $M_R \, V \, M_{R^2} V \; \ldots \; V \, M_{R^n}$
- More efficient method: Warshall's algorithm

Transitive Closure -- Example

-
- $M_R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
- $M_R \, V \; M_{R^2} V \;\; \ldots \; V \; M_{R^n}$?

Equivalence Relations

Example:

Let S = {people in this classroom}, and let

R = {(a,b): a has same # of coins in his/her bag as b}

Quiz time:

Is R reflexive? Yes

Is R symmetric? Yes

Is R transitive? Yes

This is a special kind of relation, characterized by the properties it has.

What's special about it?

Everyone with the same # of coins as you is just like you.

Equivalence Relations

Formally:

Relation R on A is an *equivalence relation* if R is

- Reflexive ($\forall\, a \in A$, aRa)
- Symmetric (aRb --> bRa)
- Transitive (aRb AND bRc --> aRc)

Recall:

aRb denotes $(a,b) \in R$.

Example:

S = Z (integers), R = $\{(a,b) : a \equiv b \bmod 4\}$

Is this relation an equivalence relation on S?

Have to PROVE reflexive, symmetric, transitive.

Equivalence Relations

Example:

$S = Z$ (integers), $R = \{(a,b) : a \equiv b \bmod 4\}$

Is this relation an equivalence relation on S?

Start by thinking of R a different way: aRb iff there is an int k so that $a = 4k + b$. Your quest becomes one of finding ks.

Let a be any integer. aRa since $a = 4 \cdot 0 + a$.

Consider aRb. Then $a = 4k + b$. But $b = -4k + a$.

Consider aRb and bRc. Then $a = 4k + b$ and $b = 4j + c$. So, $a = 4k + 4j + c = 4(k+j) + c$.

Equivalence Relations

Example:

- S = Z (integers), R = {(a,b) : a = b or a =- b}. Is this relation an equivalence relation on S?
- Have to prove reflexive, symmetric, transitive.

Equivalence Relations

Example:

- S = **R** (real numbers), R = {(a,b) : a – b is an integer}. Is this relation an equivalence relation on S?
- Have to prove reflexive, symmetric, transitive.

Equivalence Relations

Example:

- S = **R** (real numbers), R = {(a,b) : a – b is an integer}. Is this relation an equivalence relation on S?
- Have to prove reflexive, symmetric, transitive.

Equivalence Relations

- Example:
- $S = \mathbf{N}$ (natural numbers), $R = \{(a,b) : a \mid b\}$. Is this relation an equivalence relation on S?
- Have to prove reflexive, symmetric, transitive.

Equivalence Classes

Example:

Back to coins in bags.

Definition: Let R be an equivalence relation on S. The *equivalence class* of $a \in S$, $[a]_R$, is

$$[a]_R = \{b: aRb\}$$

a is just a name for the equiv class. Any member of the class is a representative.

Equivalence Classes

Definition: Let R be an equivalence relation on S. The *equivalence class* of $a \in S$, $[a]_R$, is

$[a]_R = \{b: aRb\}$

Notice this is just a subset of S.

What does the set of equivalence classes on S look like?

To answer, think about the relation from before:

S = {people in this room}

R = {(a,b) : a has the same # of coins in his/her bag as b}

In how many different equivalence classes can each person fall?

Similarly, consider the $a \equiv b \bmod 4$ relation

Today's Reading

- Rosen 9.4 and 9.5